

# On some dynamical and geometrical properties of the Maxwell-Bloch equations with a quadratic control

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## Abstract

In this paper, we analyze the stability of the real-valued Maxwell-Bloch equations with a control that depends on state variables quadratically. We also investigate the topological properties of the energy-Casimir map, as well as the existence of periodic orbits and explicitly construct the heteroclinic orbits.<sup>1</sup>

**Keywords:** Maxwell-Bloch equations, Hamiltonian systems, stability theory, energy-Casimir map, periodic orbits, heteroclinic orbits.

## 1 Introduction

The description of the interaction between laser light and a material sample composed of two-level atoms begins with Maxwell's equations of the electric field and Schrödinger's equations for the probability amplitudes of the atomic levels. The resulting dynamics is given by Maxwell - Schrödinger equations which have Hamiltonian formulation and moreover there exists a homoclinic chaos [8].

Using the Melnikov method [14], in [9] the presence of special homoclinic orbits for the dynamics of an ensemble of two-level atoms in a single-mode resonant laser cavity with external pumping and a weak coherent probe modeled by Maxwell-Bloch's equations with a probe was established.

Fordy and Holm [7] discussed the phase space geometry of the solutions of the system introduced by Holm and Kovacic [8].

In 1992, David and Holm [6] presented the phase space geometry of the mentioned system restricted to  $\mathbb{R}$ , so named real-valued Maxwell-Bloch equations:

$$\begin{cases} \dot{x} = y \\ \dot{y} = xz \\ \dot{z} = -xy \end{cases} \quad (1.1)$$

In 1996, Puta [18] considered system (1.1) with a linear, respectively a quadratic control  $u$  about  $Oy$  axis:

$$\begin{cases} \dot{x} = y \\ \dot{y} = xz + u \\ \dot{z} = -xy \end{cases} \quad (1.2)$$

These particular perturbations arise naturally in controllability context and were analyzed from the dynamical point of view. More precisely, in the case of the quadratic control  $u = (k - 1)xz$  with the parameter  $k > 0$  [18], the dynamical analysis is done by proving that the restricted dynamics on each symplectic leaf of the associated Poisson configuration manifold is equivalent to the dynamics of Duffing oscillator with control and with the pendulum dynamics.

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In our work, we consider system (1.2), where  $u = (k - 1)xz$  with  $k < 0$ . We give a Poisson structure and we find a symplectic realization of the system. Using the method introduced in [20], we find the image of the energy-Casimir map and we study the topology of the fibers of the energy-Casimir map.

For details on Poisson geometry and Hamiltonian mechanical system, see, e.g. [5], [13], [17], [12].

## 2 Poisson structure, symplectic realization and geometric pre-quantization

Considering the quadratic parametric control  $u = (k - 1)xz$ , system (1.2) becomes:

$$\begin{cases} \dot{x} = y \\ \dot{y} = kxz \\ \dot{z} = -xy \end{cases}, \quad (2.1)$$

where  $k < 0$  is the tuning parameter, according to the classification of chaos control methods [3], [4].

The constant of motion

$$H_k(x, y, z) = \frac{1}{2}(y^2 + kz^2), \quad C(x, y, z) = \frac{1}{2}x^2 + z$$

were given in [18]. Using the Euclidean space  $\mathbb{R}^3$  with a modified cross-product as Lie algebra, a Poisson structure  $\Pi$ ,

$$\Pi = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & x \\ -0 & -x & 0 \end{bmatrix}$$

was also given.

We are going to give a Lie algebra, isomorphic with that mentioned above, on its dual space the same Poisson structure is obtained.

Let us considering the Heisenberg Lie group  $H_3$ ,

$$H_3 = \{A \in GL(3, \mathbb{R}) \mid A = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, a, b, c \in \mathbb{R}\}.$$

The corresponding Lie algebra  $h_3$  is

$$h_3 = \{X \in gl(3, \mathbb{R}) \mid X = \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix}, a, b, c \in \mathbb{R}\}.$$

Note that, as a real vector space,  $h_3$  is generated by the base  $B_{h_3} = \{E_1, E_2, E_3\}$ , where

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The following bracket relations  $[E_1, E_2] = 0, [E_1, E_3] = 0, [E_2, E_3] = E_1$ , hold.

Following [12], it is easy to see that the bilinear map  $\Theta : h_3 \times h_3 \rightarrow \mathbb{R}$  given by the matrix  $(\Theta_{ij})_{1 \leq i, j \leq 3}$ ,  $\Theta_{12} = -\Theta_{21} = 1$  and 0 otherwise, is a 2-cocycle on  $h_3$  and it is not a coboundary since  $\Theta(E_1, E_2) = 1 \neq 0 = f([E_1, E_2])$ , for every linear map  $f, f : h_3 \rightarrow \mathbb{R}$ .

On the dual space  $h_3^* \simeq \mathbb{R}^3$ , a modified Lie-Poisson structure is given in coordinates by

$$\Pi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & -x & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & x \\ 0 & -x & 0 \end{bmatrix}.$$

The function  $H_k$  is the Hamiltonian and  $C$  is a Casimir of our configuration.

The next proposition states that system (2.1) can be regarded as a Hamiltonian mechanical system.

**Proposition 2.1.** *The Hamilton-Poisson mechanical system  $(\mathbb{R}^3, \Pi, H_k)$  has a full symplectic realization  $(\mathbb{R}^4, \omega, \tilde{H}_k)$ , where*

$$\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$$

and

$$\tilde{H}_k = \frac{1}{2} \left( p_1^2 + kp_2^2 - kp_2q_1^2 + \frac{k}{4}q_1^4 \right).$$

**Proof.** The corresponding Hamilton's equations are:

$$\begin{cases} \dot{q}_1 = p_1 \\ \dot{q}_2 = kp_2 - \frac{k}{2}q_1^2 \\ \dot{p}_1 = kp_2q_1 - \frac{k}{2}q_1^3 \\ \dot{p}_2 = 0. \end{cases} \quad (2.2)$$

If we define the application  $\varphi : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ ,

$$\varphi(q_1, q_2, p_1, p_2) = (x, y, z) = \left( q_1, p_1, p_2 - \frac{1}{2}q_1^2 \right),$$

then it is easy to see that  $\varphi$  is a surjective submersion, the equations (2.2) are mapped onto the equations (2.1) and the canonical structure  $\{\cdot, \cdot\}_\omega$  induced by  $\omega$  is mapped onto the Poisson structure  $\Pi$ .

Therefore  $(\mathbb{R}^4, \omega, \tilde{H}_k)$  is a full symplectic realization of the Hamilton-Poisson mechanical system  $(\mathbb{R}^3, \Pi, H_k)$ .

Now, it is naturally to ask if system (2.2) is completely integrable. The answer is given in

**Proposition 2.2.** *The Hamilton mechanical system  $(\mathbb{R}^4, \omega, \tilde{H}_k)$  given above is completely integrable.*

**Proof.** Taking into account the definition of a completely integrable Hamiltonian system [12], system (2.2) has two differentiable first integrals

$$\tilde{H}_k = \frac{1}{2} \left( p_1^2 + kp_2^2 - kp_2q_1^2 + \frac{k}{4}q_1^4 \right) \quad \text{and} \quad I = p_2$$

defined on  $\mathbb{R}^4$ , which are in involution ( $\{\tilde{H}_k, I\}_\omega = 0$ ), and whose differentials are linearly independent on the dense open subset

$$\begin{aligned} \Omega &= \{(q_1, q_2, p_1, p_2) \in \mathbb{R}^4 : \text{rank } J = 2\} \\ &= \mathbb{R}^4 \setminus \{(q_1, q_2, p_1, p_2) : p_1 = 0, q_1^3 - 2q_1p_2 = 0\} \end{aligned}$$

of  $\mathbb{R}^4$ , where  $J$  is the Jacobian matrix of  $\tilde{H}_k$  and  $I$ , which we set out to prove.

It is known that the symplectic manifold  $(\mathbb{R}^4, \omega = d\theta)$ ,  $\theta = p_1 dq_1 + p_2 dq_2$ , is quantizable from the geometric quantization point of view [22] with the Hilbert representation space  $\mathcal{H} = L^2(\mathbb{R}^4, \mathbb{C})$  and the prequantum operator  $\delta^\theta$ ,

$$\delta^\theta : f \in C^\infty(\mathbb{R}^4, \mathbb{R}) \mapsto \delta_f^\theta : \mathcal{H} \rightarrow \mathcal{H},$$

where

$$\delta_f^\theta = -i\hbar X_f - \theta(X_f) + f,$$

$\hbar$  is the Planck constant divided by  $2\pi$  and  $X_f = \sum_{k=1}^2 \left( \frac{\partial f}{\partial p_k} \cdot \frac{\partial}{\partial q_k} - \frac{\partial f}{\partial q_k} \cdot \frac{\partial}{\partial p_k} \right)$ .

We can state the following prequantization result:

**Proposition 2.3.** *The pair  $(\mathcal{H}, \delta)$ , where  $\mathcal{H} = L^2(\mathbb{R}^4, \mathbb{C})$  and*

$$\delta : F \in C^\infty(\mathbb{R}^3, \mathbb{R}) \mapsto \delta_F, \quad \delta_F = \delta_{F \circ \varphi}^\theta,$$

*gives a prequantization of the Poisson manifold  $(\mathbb{R}^3, \Pi)$ .*

**Proof.** One easily check that Dirac's conditions are all satisfied.

We note that similar results for Maxwell-Bloch equations are given in [19].

### 3 Stability of equilibria and the image of the energy-Casimir mapping

In this section we give the stability properties of the equilibrium states of (2.1) and we study the image of the energy-Casimir map  $\mathcal{E}C_k$  associated with the Hamilton-Poisson realization of system (2.1).

The equilibrium states of system (2.1) are given as the union of the following two families:

$$\mathcal{E}_k^1 = \{(M, 0, 0) \mid M \in \mathbb{R}\}$$

$$\mathcal{E}_k^2 = \{(0, 0, M) \mid M \in \mathbb{R}\}.$$

We recall [18] that the equilibrium state  $e_M = (M, 0, 0) \in \mathcal{E}_k^1$ ,  $M \neq 0$ , is unstable.

For the other equilibria we prove the following:

**Proposition 3.1.** *Let  $e_M = (0, 0, M) \in \mathcal{E}_k^2$  be an arbitrary equilibrium state. Then  $e_M$  is nonlinear stable for  $M > 0$  and unstable for  $M < 0$ .*

**Proof.** For  $M < 0$ , the eigenvalues of the characteristic polynomial associated with the linearization of system (2.1) at  $e_M = (0, 0, M)$  are  $\lambda_1 = 0$ ,  $\lambda_{2,3} = \pm \sqrt{kM} \in \mathbb{R}$ , hence  $e_M$  is unstable.

To study the nonlinear stability of the equilibria from  $\mathcal{E}_k^2$  in the case  $M > 0$  we are using the Arnold stability test [1], [10]. To do that, let  $F_\lambda \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$  be defined by  $F_\lambda = H_k - \lambda C = \frac{1}{2}(y^2 + kz^2) - \lambda \left( \frac{1}{2}x^2 + z \right)$ , where  $\lambda$  is a real parameter. Then, we have successively the following:

- (i)  $dF_\lambda(0, 0, M) = 0$  if and only if  $\lambda = kM$ ;
- (ii)  $W = \ker dC(M, 0, k) = \text{Sp}_{\mathbb{R}} \{(1, 0, 0), (0, 1, 0)\}$ ;
- (iii)  $d^2 F_{\lambda=kM}(0, 0, M)|_{W \times W} = -kM dx^2 + dy^2$  is positive definite for  $M > 0$ .

Hence, from the Arnold stability test we conclude that all the equilibria from  $\mathcal{E}_k^2$  with  $M > 0$  are nonlinear stable.

Now, let us consider the energy-Casimir mapping  $\mathcal{E}C_k \in C^\infty(\mathbb{R}^3, \mathbb{R}^2)$ , given by

$$\mathcal{E}C_k(x, y, z) = (H_k(x, y, z), C(x, y, z)) = \left( \frac{1}{2}(y^2 + kz^2), \frac{1}{2}x^2 + z \right).$$

The next proposition gives a characterization of the image of the energy-Casimir map  $\mathcal{E}C_k$ .

**Proposition 3.2.** *The image of the energy-Casimir map is*

$$\text{Im}(\mathcal{E}C_k) = S_{k,-}^I \cup S_{k,-}^{II} \cup S_{k,-}^{III},$$

where

$$\begin{aligned} S_{k,-}^I &= \left\{ (h, c) \in \mathbb{R}^2 : c^2 \leq \frac{2h}{k} \right\} \\ S_{k,-}^{II} &= \left\{ (h, c) \in \mathbb{R}^2 : c^2 \geq \frac{2h}{k}, h \leq 0, c \geq 0 \right\} \\ S_{k,-}^{III} &= \mathbb{R}^2 \setminus \text{Int} (S_{k,-}^I \cup S_{k,-}^{II}). \end{aligned}$$

( $\text{Int}(A)$  = the interior of the set  $A$ ).

**Proof.** The conclusion follows by algebraic computations using the definition of the energy-Casimir map.

As any semialgebraic manifold has a canonical Whitney stratification [16], we will describe it by using the image through the energy-Casimir map of subsets of the families of equilibria of system (2.1).

**Proposition 3.3.** *The semialgebraic canonical Whitney stratifications of  $S_{k,-}^I$ ,  $S_{k,-}^{II}$ ,  $S_{k,-}^{III}$  are given by the following:*

- (i)  $S_{k,-}^I = \text{Im}(\mathcal{E}C_k|_{\mathcal{E}_k^2}) \cup \Sigma^p(S_{k,-}^I)$ , where  $\Sigma^p(S_{k,-}^I) = \{(h, c) \in \mathbb{R}^2 : c^2 < \frac{2h}{k}\}$  is the principal stratum.
- (ii)  $S_{k,-}^{II} = \text{Im}(\mathcal{E}C_k|_{\mathcal{E}_k^{2,s}}) \cup \text{Im}(\mathcal{E}C_k|_{\mathcal{E}_k^{1,u} \cap \mathcal{E}_k^{2,u}}) \cup \text{Im}(\mathcal{E}C_k|_{\mathcal{E}_k^{1,u}}) \cup \Sigma^p(S_{k,-}^{II})$ , where  $\Sigma^p(S_{k,-}^{II}) = \left\{ (h, c) \in \mathbb{R}^2 : c^2 > \frac{2h}{c}, h < 0, c > 0 \right\}$  is the principal stratum.
- (iii)  $S_{k,-}^{III} = \text{Im}(\mathcal{E}C_k|_{\mathcal{E}_k^{2,u}}) \cup \text{Im}(\mathcal{E}C_k|_{\mathcal{E}_k^{1,u} \cap \mathcal{E}_k^{2,u}}) \cup \text{Im}(\mathcal{E}C_k|_{\mathcal{E}_k^{1,u}}) \cup \Sigma^p(S_{k,-}^{III})$ , where the principal stratum  $\Sigma^p(S_{k,-}^{III})$  is the interior of  $S_{k,-}^{III}$ .

All the stratification results can be gathered as shown in Figure 1, where we denoted  $\text{Im}(\mathcal{E}C_k|_{\mathcal{E}_k^i})$  by  $\Sigma_{k,-}^i$ , and superscripts "u" and "s" stand respectively for "unstable" and "stable".

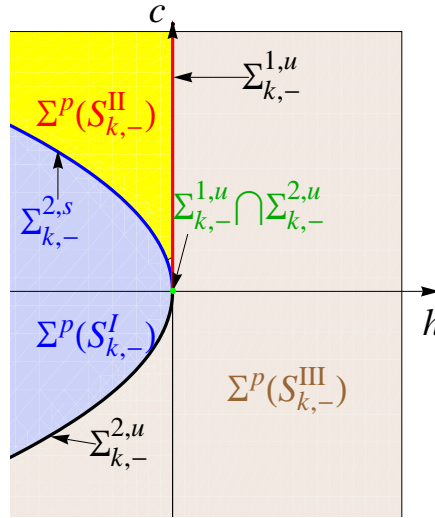


Figure 1: Whitney canonical stratification of semialgebraic manifolds  $S_{k,-}^I$ ,  $S_{k,-}^{II}$ ,  $S_{k,-}^{III}$

## 4 The topology of the fibers of the energy-Casimir map

In this section the topology of the fibers of  $\mathcal{E}C_k$  is described.

A fiber of the map  $\mathcal{E}C_k : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is the preimage of an element  $(h, c) \in \mathbb{R}^2$  through  $\mathcal{E}C_k$ , that is  $\mathcal{F}_{(h,c)} = \{(x, y, z) \in \mathbb{R}^3 : \mathcal{E}C_k(x, y, z) = (h, c)\}$ .

**Proposition 4.1.** *The classification of the fibers  $\mathcal{F}_{(h,c)}$  with  $(h, c)$  belonging to the strata of  $\text{Im}(\mathcal{E}C_k)$  can be described as follows:*

- (i) If  $(h, c) \in \Sigma_{k,-}^{2,s}$  then  $\mathcal{F}_{(h,c)} = \{(0, 0, c) : c > 0\} \cup \left\{ (x(t), y(t), z(t)) : t \in \left(-\frac{\pi}{2\sqrt{-kM}}, \frac{\pi}{2\sqrt{-kM}}\right) \right\} \cup$

$\cup \left\{ (-x(t), -y(t), z(t)) : t \in \left( -\frac{\pi}{2\sqrt{-kM}}, \frac{\pi}{2\sqrt{-kM}} \right) \right\}$  (Figure 2), where

$$\begin{aligned} x(t) &= 2\sqrt{M} \sec(\sqrt{-kM}t) \\ y(t) &= 2M\sqrt{-k} \sec(\sqrt{-kM}t) \cdot \tan(\sqrt{-kM}t) \\ z(t) &= -M[1 + 2 \tan^2(\sqrt{-kM}t)]. \end{aligned}$$

(ii) If  $(h, c) \in \Sigma_{k,-}^{2,u}$  then  $\mathcal{F}_{(h,c)} = \{(x(t), y(t), z(t)) : t \in (0, \infty)\} \cup \{(x(t), y(t), z(t)) : t \in (-\infty, 0)\} \cup \{(-x(t), -y(t), z(t)) : t \in (0, \infty)\} \cup \{(-x(t), -y(t), z(t)) : t \in (-\infty, 0)\}$  (Figure 3), where

$$\begin{aligned} x(t) &= \frac{4\sqrt{Mp(t)}}{p(t) + 1} \\ y(t) &= \frac{4M\sqrt{kp(t)}(p(t) - 1)}{(p(t) + 1)^2} \\ z(t) &= M \cdot \frac{p^2(t) - 6p(t) + 1}{(p(t) + 1)^2} \end{aligned}$$

with  $p(t) = -e^{2\sqrt{kM}t}$ ,

(iii) If  $(h, c) \in \Sigma_{k,-}^{1,u}$  then  $\mathcal{F}_{(h,c)}$  is a pair of heteroclinic orbits (Figure 4);

(iv) If  $(h, c) \in \Sigma^P(S_{k,-}^I)$  or  $(h, c) \in \Sigma^P(S_{k,-}^{II})$  then

$\mathcal{F}_{(h,c)} = \{(x(t), y(t), z(t)) : t \in (0, t_f)\} \cup \{(-x(t), y(t), z(t)) : t \in (0, t_f)\} \cup \{(x(t), -y(t), z(t)) : t \in (0, t_f)\} \cup \{(-x(t), -y(t), z(t)) : t \in (0, t_f)\}$ , where

$$\begin{aligned} x &= \sqrt{2(c - z)} \\ y &= \sqrt{2h - kz^2} \\ z &= \frac{2}{k}\mathcal{P} + \frac{c}{3} \end{aligned}$$

where  $\mathcal{P}$  is the Weierstrass elliptic function with the invariants  $g_2 = \frac{k^2}{3}c^2 + 2kh$ ,  $g_3 = \frac{k^3}{27}c^3 - \frac{2k^2}{3}hc$  (for details on elliptic functions see [11]);

(v) If  $(h, c) \in \Sigma^P(S_{k,-}^{II})$  then  $\mathcal{F}_{(h,c)}$  consists from the same four curves like as (iv) and a periodic orbit (Figure 5).

**Proof.** (i) Solving the system  $(H_k(x, y, z), C(x, y, z)) = (h, c)$ , where  $(h, c) \in \Sigma_{k,-}^{2,s}$  it results a solution  $(0, 0, c)$ ,  $c > 0$  and a curve given by the intersection of surfaces  $y^2 + kz^2 = 2h$  and  $x^2 + 2z = 2c$  (Figure 2).

Considering the above surfaces as constants of motion we first reduce system (2.1) from three degrees of freedom to one degree of freedom and then integrate the resulting reduced differential equation, it follows the conclusion.

(ii)-(v) It results using the same method with  $(h, c)$  from other stratum.

In the following we study the existence of periodic orbits of system (2.1) around nonlinear stable equilibrium states.

Since the linearized system around nonlinear stable equilibrium states has a zero eigenvalue, we cannot apply Weinstein's result [21] or Moser's result [15] for proving the existence of periodic orbits. Then, we will apply Theorem 2.1 from [2], which ensures the existence of periodic orbits around an equilibrium point. We recall this result:

**Theorem.** Let  $\dot{x} = X(x)$  be a dynamical system,  $x_0$  an equilibrium point, i.e.,  $X(x_0) = 0$  and  $C := (C_1, \dots, C_k) : M \rightarrow \mathbb{R}^k$  a vector valued constant of motion for the above dynamical system with  $C(x_0)$  a regular value for  $C$ . If

(i) the eigenspace corresponding to the eigenvalue zero of the linearized system around  $x_0$  has dimension

$k$ ,

(ii)  $DX(x_0)$  has a pair of pure complex eigenvalues  $\pm i\omega$  with  $\omega \neq 0$ ,

(iii) there exist a constant of motion  $I : M \rightarrow \mathbb{R}$  for the vector field  $X$  with  $dI(x_0) = 0$  and such that  $d^2I(x_0)|_{W \times W} > 0$ , where  $W = \bigcap_{i=1}^k \ker dC_i(x_0)$ ,

then for each sufficiently small  $\varepsilon \in \mathbb{R}$ , any integral surface  $I(x) = I(x_0) + \varepsilon^2$  contains at least one periodic solution of  $X$  whose period is close to the period of the corresponding linear system around  $x_0$ ."

In our case, for  $M > 0$ , the eigenvalues of the characteristic polynomial associated with the linearization of system (2.1) at  $(0, 0, M)$  are  $\lambda_1 = 0$ ,  $\lambda_{2,3} = \pm i\sqrt{-kM}$  and the eigenspace corresponding to the eigenvalue zero has dimension 1. Considering the constant of motion  $I : \mathbb{R}^3 \rightarrow \mathbb{R}$ ,

$$I(x, y, z) = -kMx^2 + y^2 + k(z - M)^2,$$

it follows that  $dI(0, 0, M) = 0$  and  $d^2I(0, 0, M)|_{W \times W} > 0$ , where

$$W = \ker dC(0, 0, M) = \text{Span}_{\mathbb{R}} \{(1, 0, 0), (0, 1, 0)\}.$$

By applying Theorem 2.1 from [2] we have proved the following result:

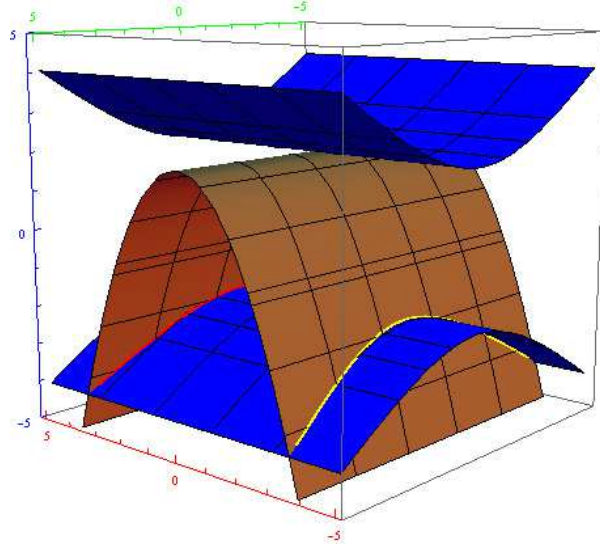


Figure 2: The fiber  $\mathcal{F}_{(h,c)}, (h,c) \in \Sigma_{k,-}^{2,s}$ , consists of a stable equilibrium point and two curves, as intersection of two surfaces

**Proposition 4.2.** Let  $e_M = (0, 0, M) \in \mathcal{E}_k^{2,s}$  be such that  $M > 0$ . Then for each sufficiently small  $\varepsilon \in \mathbb{R}_+^*$ , any integral surface

$$\Sigma_{\varepsilon}^{e_M} : -kMx^2 + y^2 + k(z - M)^2 = \varepsilon^2$$

contains at least one periodic orbit  $\gamma_{\varepsilon}^{e_M}$  of system (2.1) whose period is close to  $\frac{2\pi}{\sqrt{-kM}}$ .

In the sequel the heteroclinic orbits are given.

To obtain a parametric form of the heteroclinic orbits, we first reduce (2.1) from three degrees of freedom to one degree of freedom by using the relations obtained by eliminating  $x$  and  $y$  from  $H_k(x, y, z) = H_k(0, 0, M)$  and  $C(x, y, z) = C(0, 0, M)$  and then integrate the resulting reduced differential equation. We find  $z$  and then  $x$  and  $y$ . Thus, we obtain the following result:

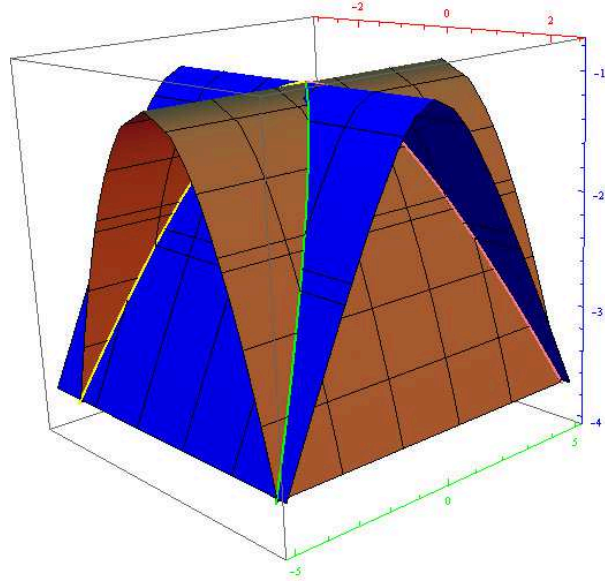


Figure 3: The fiber  $\mathcal{F}_{(h,c)}, (h,c) \in \Sigma_{k,-}^{2,u}$ , consists of four curves, as intersection of two surfaces

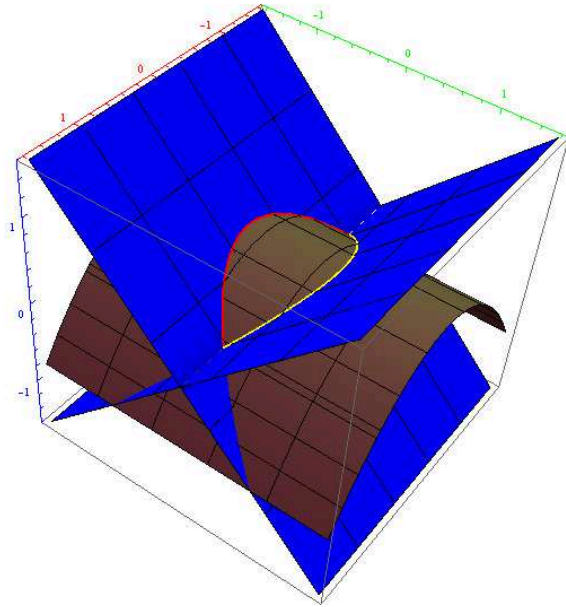


Figure 4: The fiber  $\mathcal{F}_{(h,c)}, (h,c) \in \Sigma_{k,-}^{1,u}$ , consists of pair of heteroclinic orbits, as intersection of two surfaces



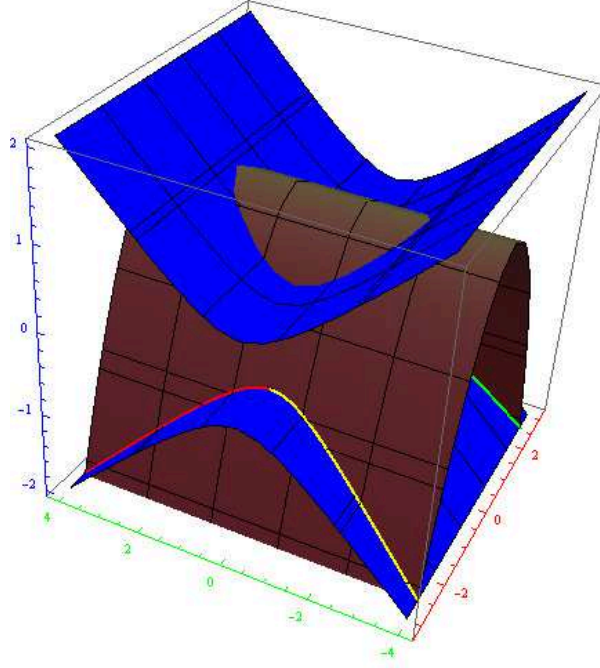


Figure 5: The fiber  $\mathcal{F}_{(h,c)}$ ,  $(h,c) \in \Sigma^p(S_{k,-}^{II})$ , consists of a periodic orbit and four curves, as intersection of two surfaces

**Proposition 4.3.** *The parametrizations of the heteroclinic orbits connecting unstable equilibria  $(-M, 0, 0)$  and  $(M, 0, 0)$ ,  $M \neq 0$ , (Figure 4), are*

$$\begin{aligned}\mathcal{H}_{(+,+,+)}^{(\pm M, 0, 0)}(t) &:= (x(t), y(t), z(t)), \\ \mathcal{H}_{(-,-,+)}^{(\pm M, 0, 0)}(t) &:= (-x(t), -y(t), z(t)),\end{aligned}$$

where

$$\begin{aligned}x(t) &= M \cdot \frac{p(t) - 1}{p(t) + 1} \\ y(t) &= 2M^2 \sqrt{-k} \cdot \frac{p(t)}{[p(t) + 1]^2} \\ z(t) &= 2M^2 \cdot \frac{p(t)}{[p(t) + 1]^2}\end{aligned}$$

with  $p(t) = e^{M\sqrt{-k}(t+\alpha)}$ ,  $t \in \mathbb{R}$  and  $\alpha \in \mathbb{R}$  is an arbitrary real constant.

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## References

- [1] V. Arnold, Conditions for nonlinear stability of stationary plane curvilinear flows on an ideal fluid, Akad. Nauk. Doklady SSSR, **162** (1965) 773–777.

- [2] P. Birtea, M. Puta, R.M.Tudoran, Periodic orbits in the case of zero eigenvalue, C.R. Acad. Sci. Paris, Ser. I, **344** (2007) 779–784.
- [3] G. Chen, X.Dong, From chaos to order–Perspectives and methodologies in controlling chaotic non-linear dynamic systems, Int. J. of Bifurcation and Chaos, Vol. **3** (1993) 1363-1409.
- [4] G. Chen, X.Dong, From Chaos to Order: Methodologies, Perspectives and Applications, World Scientific Pub. Co., Singapore, 1998.
- [5] R.H. Cushman, L. Bates, Global Aspects of Classical Integrable Systems, Birkhäuser, Basel, 1997.
- [6] D. David, D.D. Holm, Multiple Lie-Poisson Structures, Reductions, and Geometric Phases for the Maxwell-Bloch Travelling Wave Equations, J. Nonlinear Sci., Vol. **2** (1992) 241-262.
- [7] A. Fordy, D.D. Holm, A tri-Hamiltonian formulation of the self-induced transparency equations, Phys. Lett. A, **160**(1991) 143-148.
- [8] D.D. Holm, G. Kovacic, Homoclinic chaos in a laser matter system, Physica D., Vol. **56** (1992) 270–300.
- [9] D.D. Holm, G. Kovacic, T.A. Wettergren, Homoclinic orbits in the Maxwell-Bloch equations with a probe, Phys. Rev. E, **54** (1996) 243–256.
- [10] D.D. Holm, J. Marsden, T.S. Rațiu, A. Weinstein, Nonlinear stability of fluid and plasma equilibria, Physics Reports, vol. **123**, no. 2 (1985) 1–116.
- [11] D.F. Lawden, Elliptic Functions and Applications, Springer–Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, Hong Kong, 1989.
- [12] P. Libermann, C.-M. Marle, Symplectic Geometry and Analytical Mechanics, D. Reidel, Dordrecht, 1987.
- [13] J. Marsden, T.S. Rațiu, Introduction to Mechanics and Symmetry, 2nd Ed. Text and Appl. Math. **17**, Springer, Berlin, 1999.
- [14] V.K.Melnikov, On the stability of the center for time periodic perturbations, Trans. Moscow Math., **12** (1963) 1–57.
- [15] J. Moser, Periodic orbits and a theorem by Alan Weinstein, Comm. Pure Appl. Math., **29** (1976) 727–747.
- [16] M.J. Pflaum, Analytic and Geometric Study of Stratified Spaces, Lecture Notes in Math., **510**, Springer, Berlin, 2001.
- [17] M. Puta, Hamiltonian mechanical system and geometric quantization, Kluwer Academic Publishers, Dordrecht, Boston, London, 1993.
- [18] M. Puta, Three-dimensional real-valued Maxwell-Bloch equations with controls, Reports on Mathematical Physics, Volume **37**, Issue **3** (1996) 337–348.
- [19] M. Puta, Integrability and geometric prequantization of the Maxwell-Bloch equations, Bull. Sci. math, Volume **122** (1998) 243–250.
- [20] R.M. Tudoran, A. Aron, Ș. Nicoară, On a Hamiltonian Version of the Rikitake System, SIAM J. Applied Dynamical Systems, Vol. **8**, No. **1** (2009) 454–479.
- [21] A. Weinstein, Normal modes for non-linear Hamiltonian systems, Invent. Math., **20** (1973) 47–57.
- [22] N.M.J. Woodhouse, Geometric quantization, Oxford University Press, 1990.